## GEOMETRY OF THE CONSTRAINTS MANIFOLD

### PRINCIPAL DIRECTIONS AND CORRESPONDING NORMAL CURVATURES

The principal normal to an arbitrary curve on a manifold is not, in general, perpendicular to the manifold and can be decomposed into a part perpendicular to the manifold and a projection onto the tangent plane to the manifold. Geodesics however seem to have their principal normal perpendicular to the surface. This is true for a surface embedded in a three-dimensional space and it holds for the constrained geodesics of this paper as well. This facilitates investigating the *principal directions* [the directions along which the normal curvature is extremum], and the corresponding *principal normal curvatures* [the corresponding curvatures] of the manifold since the curvature of any constrained geodesic passing through a point in a given direction is also the normal curvature of the manifold in that direction. We recall the equation of the constrained geodesic

$$u^i_{ss} + \Gamma^i_{jk}\, u^j_s u^k_s = -\alpha_p \theta^i_p \tag{1}$$

where

$$\theta^i_p = g^{ik} H^p_k \tag{2}$$

For an arbitrary curve its principal normal **n** and curvature $\kappa$ are defined as

$$\kappa n^i = u^i_{ss} + \Gamma^i_{jk}\, u^j_s u^k_s \tag{3}$$

This is the tensorially correct definition of curvature for a non-euclidean geometry. It follows that the curvature of an arbitrary curve is equal to

$$\kappa^2 = [\, u^i_{ss} + \Gamma^i_{jk}\, u^j_s u^k_s \,]\,[\, u^i_{ss} + \Gamma^i_{mn}\, u^m_s u^n_s \,] \tag{4}$$

For the constrained geodesic the principal normal can be decomposed entirely in terms of the normals to the individual constraint surfaces and is therefore perpendicular to the manifold. In addition its curvature is

$$\kappa^2 = \alpha_p \alpha_q\, \theta^k_p \theta^k_q \tag{5}$$

Substituting the solution of the Gram system we obtain the alternative form

$$\kappa^2 = h^{pq} \beta_p \beta_q \tag{6}$$

where the right hand side is as before

$$\beta_p = (H^p_{kl} - \Gamma^n_{kl} H^p_n)\, u^k_s u^l_s \tag{7}$$

The compact expression of the curvature using the quadratic form involving the $\beta$ (6) looks very attractive. It would be possible to generate a basis of n-m vectors for the feasible search directions in the orthogonal complement of the normals to the constraints surfaces. However, each of the m eigenvectors could be decomposed in many ways into the n-m transforms of these base vectors if m<n/2 and for m>n/2 a least squares methods would have to be used. This leads us to consider the ratio of quadratic forms

$$\beta_{\tilde{p}}(p) = \frac{(H^p_{kl} - \Gamma^n_{kl} H^p_n)\, p^k p^l}{g_{kl}\, p^k p^k} \quad \overset{Euclidean}{=} \quad \frac{H^p_{kl}\, p^k p^l}{p^k p^k} \tag{8}$$

The principal directions at a point are the directions along which the normal curvature is stationary and the principal curvatures are the corresponding extremal values, and these are therefore the stationary points of the function

$$Q^2(p) = h^{pq} \frac{(H^p_{kl}\, p^k p^l)}{(g_{kl}\, p^k p^k)} \frac{(H^q_{mn} p^m p^n)}{(g_{mn}\, p^m p^n)} \tag{9}$$

whereby the direction is subject to the tangent plane constraints (9)

$$H^p_k\, p^k = 0 \qquad\qquad (p = 1 \cdots m) \tag{10}$$

Note that the inverse Gram matrix pre-multiplying (9) depends only on the position of the point but not on the initial direction of the geodesic and is therefore invariant as regards to the problem of finding the extremal directions.

The case of a *single* constraint is special since in this case the quadratic form (9) reduces to a single term

$$Q^2 = \frac{1}{(g^{kl} H_k H_l)} \left[ \frac{(H_{kl}\, p^k p^l)}{(g_{kl}\, p^k p^l)} \right]^2 \tag{11}$$

In view of the first term in (11) being independent of the search direction (although dependent on the initial point ***P***) and the remainder being a square minimization of (11) is equivalent to minimizing the function

$$Q(p) = \frac{1}{\sqrt{(g^{kl} H_k H_l)}} \frac{(H_{kl}\, p^k p^l)}{(g_{kl}\, p^k p^l)} \tag{12}$$

subject to the single constraint which is a [generalized] eigenvalue problem for the Hessian of the single constraint using the metric of the embedding space subject to an auxiliary constraint.

The special case of a single constraint is known as a *hypersurface* and it is clear from comparing equations (9) and (12) that the hypersurface is a special case which allows considerable simplification.

The main subject of classical differential geometry is the geometry of surface in a three-dimensional euclidean space. This is the simplest case of a hypersurface where the surface is the single constraint. In the classical differential geometry of a surface it is shown that the principal directions and corresponding curvatures are the extremal values of the ratio of the so-called second and the first fundamental forms for the surface - expressed in terms of the *two* coordinates of the surface. It can be shown using some algebra that the ratio of quadratic forms thus defined is equivalent to equation (11) subject to the condition that the direction lies within the tangent plane as implied by equation (10). Therefore, equations (11) and (10) which define the principal directions and principal curvatures of a surface are equivalent to the corresponding equations of classical differential geometry but are constructed in terms of the coordinates of the embedding (three-dimensional) space and do not require the selection of a coordinate system in the surface. This shows that the current theory allows an approach from an entirely different angle which sometimes may be more fundamental.

In the case of multiple constraints, the minimization problem for the principal curvatures consists of minimizing a quadratic form whose terms consist of generalized eigenvalue problems. This is a considerable more difficult problem than the case of a hypersurface [a single constraint] where the solution of a “normal” generalized eigenvalue problem is required. In classical differential geometry this generalized eigenvalue problem may be solved explicitly because the order is only two which allows explicit computation of the eigenvalues.

It should finally be mentioned that the biquadratic form involves the fourth order tensor ***T*** defined as

$$T_{ijkl} = h^{pq}\left[(H^p_{ij} - \Gamma^m_{ij} H^p_m)\,(H^q_{kl} - \Gamma^n_{kl} H^q_n)\right] \tag{13}$$

It is straightforward to show that equation (12) represents the normal curvature for *any* curve *C* whose initial conditions satisfy equations (9) and (10), not necessarily a geodesic or even a curve within the manifold since derivative conditions higher than the second order required to honour the manifold have no bearing on the result. This proves that as is the case in the classical theory the normal curvature of a manifold is a function of the direction in the manifold and not of any particular curve [and that all curves having the same direction at a point *P* have the same normal curvature].

**RIEMANN TENSOR**

The nonhomogeneous terms in the right hand side of the geodesic equation (5) may be regarded as modified Christoffel symbols $\bar{\Gamma}^i_{jk}$ defined as

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + h^{pq}\,(g^{il} H^p_l)\,(H^q_{jk} - \Gamma^n_{jk} H^q_n) \tag{14}$$

The modified Christoffel symbols consist of the original Christoffel symbols for the embedding space augmented by an additional term which is also symmetric in the two lower indices. The additional term consists of the inverse Gram matrix and the product of two indexed terms which are obviously tensors and therefore obey the transformation law for Christoffel symbols (Appendix I).

The modified Christoffel symbols may be associated with the *n-m* dimensional manifold formed by the union of the constraints (4) wherever it exists. This permits the computation of the Riemann curvature tensor and the curvature invariant for the manifold without reference to the metric of the manifold. This computation is presented here for the holonomic case.

The *Riemann* tensor expressed in terms of the Christoffel symbols is

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml} \tag{15}$$

Substituting the modified Christoffel symbols into the Riemann tensor and lowering the contravariant index using the covariant metric tensor [for the embedding space], the Riemann tensor for the manifold may be written with all four indices covariant as

$$\bar{R}_{ijkl} = R_{ijkl} - h^{pq} H_i^p (R^n_{jkl} H_n^q) + h^{rs} \begin{bmatrix} + (H_{ik}^r - \Gamma_{ik}^u H_u^r)(H_{jl}^s - \Gamma_{jl}^v H_v^s) \\ - (H_{il}^r - \Gamma_{il}^u H_u^r)(H_{jk}^s - \Gamma_{jk}^v H_v^s) \end{bmatrix}$$
$$- h^{pq} h^{rs} H_i^p (g^{mn} H_n^q) \begin{bmatrix} + (H_{mk}^r - \Gamma_{mk}^u H_u^r)(H_{jl}^s - \Gamma_{jl}^v H_v^s) \\ - (H_{ml}^r - \Gamma_{ml}^u H_u^r)(H_{jk}^s - \Gamma_{jk}^v H_v^s) \end{bmatrix} \quad (16)$$

It is easily verified that the quantities thus defined are indeed a *tensor* of the implied type by (I) using the tensor character of the Riemann tensor for the embedding space and (ii) observing the invariant character of the inverse Gram matrices under transformation.

It should be noted that the lowering of the leading contravariant index using the covariant metric tensor for the embedding space is in the interest of legibility and not to imply that the tensor thus obtained would be the Riemann tensor for the manifold with all four indices covariant - which we cannot obtain without the metric tensor for the manifold.

The Riemann tensor may be written in a slightly different form by combining the third and fourth terms in the right hand side of (16) using the inverse Gram matrix (7) giving

$$\bar{R}_{ijkl} = R_{ijkl} - h^{pq} H_i^p (R^n_{jkl} H_n^q)$$
$$+ h^{pq} h^{rs} (g^{mn} H_m^p H_n^q) \begin{bmatrix} + (H_{ik}^r - \Gamma_{ik}^u H_u^r)(H_{jl}^s - \Gamma_{jl}^v H_v^s) \\ - (H_{il}^r - \Gamma_{il}^u H_u^r)(H_{jk}^s - \Gamma_{jk}^v H_v^s) \end{bmatrix} \quad (17)$$
$$- h^{pq} h^{rs} (g^{mn} H_i^p H_n^q) \begin{bmatrix} + (H_{mk}^r - \Gamma_{mk}^u H_u^r)(H_{jl}^s - \Gamma_{jl}^v H_v^s) \\ - (H_{ml}^r - \Gamma_{ml}^u H_u^r)(H_{jk}^s - \Gamma_{jk}^v H_v^s) \end{bmatrix}$$

We observe that the Riemann tensor for the manifold consists of the Riemann tensor for the embedding space to which have been added a term representing interactions between the Riemann tensor of the embedding space and the constraints, and a combination of two terms involving absolute second order derivatives of the constraints obtained with the aid of the Christoffel symbols of the embedding space.

The Riemann tensor for the manifold is antisymmetric in the third and the fourth indices, but, unlike a Riemann tensor deriving from a metric, not in the first pair of indices. These and other properties of the Riemann tensor in the general case of a space with connection coefficients [Christoffel symbols] but without a metric are discussed in [4]. The term involving the Riemann tensor of the embedding space may alternatively be expressed in terms of covariant derivatives of the constraints using the fundamental property of the Riemann tensor as

$$h^{pq} H_i^p (R^n_{jkl} H_n^q) = h^{pq} H_i^p (H^q_{j,kl} - H^q_{j,lk}) \quad (18)$$

If the embedding $n$-dimensional space is Euclidean, its Riemann tensor vanishes so that the first and second terms in the right hand side of (17) vanish.

If the embedding space is Euclidean the curvature tensor for the manifold reduces to

$$\boldsymbol{R}_{ijkl} = h^{rs}(H^r_{ik}H^s_{jl} - H^r_{il}H^s_{jk}) - h^{pq}h^{rs}H^p_i H^q_m (H^r_{mk}H^s_{jl} - H^r_{ml}H^s_{jk}) \tag{19}$$

which in view of (7) may also be written as

$$\begin{aligned}\boldsymbol{R}_{ijkl} = \; & h^{pq}h^{rs}(H^p_m H^q_m)(H^r_{ik}H^s_{jl} - H^r_{il}H^s_{jk}) \\ & - h^{pq}h^{rs}(H^p_i H^q_m)(H^r_{mk}H^s_{jl} - H^r_{ml}H^s_{jk})\end{aligned} \tag{20}$$

The *Ricci* tensor for the manifold is obtained in covariant form by contracting the Riemann tensor on the first and the fourth indices giving

$$\begin{aligned}\bar{\boldsymbol{R}}_{ij} = \; & \boldsymbol{R}_{ij} - h^{pq}(g^{kl}H^p_l)(R^{\;n}_{ijk}H^q_n) \\ & - h^{rs}\left[g^{kl} - h^{pq}(g^{km}H^p_m)(g^{ln}H^q_n)\right]\begin{bmatrix}(H^r_{ij} - \Gamma^u_{ij}H^r_u)(H^s_{kl} - \Gamma^v_{kl}H^s_v) \\ -(H^r_{ik} - \Gamma^u_{ik}H^r_u)(H^s_{jl} - \Gamma^v_{jl}H^s_v)\end{bmatrix}\end{aligned} \tag{21}$$

which may also be written using (7) as

$$\begin{aligned}\bar{\boldsymbol{R}}_{ij} = \; & \boldsymbol{R}_{ij} - h^{pq}(g^{kl}H^p_l)(R^{\;n}_{ijk}H^q_n) \\ & - h^{pq}h^{rs}\left[g^{kl}g^{mn}(H^p_m H^q_n) - (g^{km}H^p_m)(g^{ln}H^q_n)\right]\begin{bmatrix}(H^r_{ij} - \Gamma^u_{ij}H^r_u)(H^s_{kl} - \Gamma^v_{kl}H^s_v) \\ -(H^r_{ik} - \Gamma^u_{ik}H^r_u)(H^s_{jl} - \Gamma^v_{jl}H^s_v)\end{bmatrix}\end{aligned} \tag{22}$$

The *curvature invariant* for the manifold is obtained as usual by raising the index in the covariant Ricci tensor and contracting, giving

$$\begin{aligned}\bar{\boldsymbol{R}} = \; & \boldsymbol{R} - h^{pq}(g^{kl}H^p_l)(g^{ij}R^{\;n}_{ijk}H^q_n) \\ & - h^{pq}h^{rs}g^{ij}\left[g^{kl}g^{mn}(H^p_m H^q_n) - (g^{km}H^p_m)(g^{ln}H^q_n)\right]\begin{bmatrix}(H^r_{ij} - \Gamma^u_{ij}H^r_u)(H^s_{kl} - \Gamma^v_{kl}H^s_v) \\ -(H^r_{ik} - \Gamma^u_{ik}H^r_u)(H^s_{jl} - \Gamma^v_{jl}H^s_v)\end{bmatrix}\end{aligned} \tag{23}$$

In the case where the embedding space is Euclidean, the terms involving the curvature tensor of the embedding space vanish and $\boldsymbol{R} = \boldsymbol{0}$ so that the curvature invariant for the manifold reduces to

$$\bar{\boldsymbol{R}} = h^{rs}(H^r_{kl}H^s_{kl} - H^r_{kk}H^s_{ll}) + h^{pq}(H^p_k H^q_l)h^{rs}(H^r_{kl}H^s_{mm} - H^r_{km}H^s_{lm}) \tag{24}$$

which may also be written in an alternative form using (7) as

$$\bar{R} = h^{pq} h^{rs} \left[ (H^p_m H^q_m)(H^r_{kl} H^s_{kl} - H^r_{kk} H^s_{ll}) + (H^p_k H^q_l)(H^r_{kl} H^s_{mm} - H^r_{km} H^s_{lm}) \right] \tag{25}$$

The Gaussian curvature for the manifold is one half times the curvature invariant

$$\bar{K} = \frac{1}{2} \bar{R} \tag{26}$$

This completes the usual treatment of the curvature tensor and the resulting curvature.

In summary, the construction of the geodesic for an *n-m* dimensional manifold defined by a set of *m* algebraic equality constraints embedded in an *n*-dimensional space has led to the Christoffel symbols, the Riemann curvature tensor, and the curvature invariant for the manifold. In addition we have constructed the minimization problem satisfied by the principal directions and the corresponding principal curvatures. It has been shown that the case of a single constraint is a special case allowing simplification of the minimization function. The equivalence with the classical differential geometry of a surface in three-dimensional Cartesian space has been demonstrated in an elegant way. The construction is in terms of the coordinates of the embedding space.

## VARIATION OF A SCALAR FUNCTION ALONG A CONSTRAINED GEODESIC

It is natural to seek the solution of constrained optimization problems by considering the behaviour of their objective function along a geodesic in the manifold formed by the constraints since such a geodesic will automatically honour the constraints.

Consider the Taylor's series expansion of a curve $\boldsymbol{C}$ at a point $\boldsymbol{P}$ given by $u^i = u^i(0) = u_0^i$ in terms of arclength $s$

$$u^i(s) = u^i(0) + u_s^i(0)\,s + u_{ss}^i(0)\,\frac{s^2}{2!} + \cdots \tag{1}$$

Likewise, consider the Taylor's series expansion of an arbitrary scalar function $\boldsymbol{F}$ along a curve $\boldsymbol{C}$

$$F(s) = F(0) + F_s(0)\,s + F_{ss}(0)\,\frac{s^2}{2!} + \cdots \tag{2}$$

The derivatives of $\boldsymbol{F}$ with respect to arclength may be expressed in terms of its derivatives with respect to the coordinate functions and the coordinate functions representing $\boldsymbol{C}$ as

$$F_s = F_k\,u_s^k \tag{3}$$

$$F_{ss} = F_{kl}\,u_s^k u_s^l + F_k\,u_{ss}^k \tag{4}$$

$$F_{sss} = F_{klm}\,u_s^k u_s^l u_s^m + 3\,F_{kl}\,u_{ss}^k u_s^l + F_k\,u_{sss}^k \tag{5}$$

If $\boldsymbol{C}$ is a geodesic or a constrained geodesic, the second and higher order derivatives of the coordinate functions are governed by the ordinary differential equations for the geodesic (3) or the constrained geodesic (5) respectively.

We can use the notation for the ordinary geodesic bearing in mind that for the constrained geodesic we must replace the usual Christoffel symbols with the modified Christoffel symbols, therefore

$$u_{ss}^i = -\Gamma_{jk}^i\,u_s^j u_s^k \tag{6}$$

$$u_{sss}^i = \left[2\,\Gamma_{jn}^i\,\Gamma_{kl}^n - \frac{\partial\Gamma_{jk}^i}{\partial x^l}\right] u_s^j u_s^k u_s^l \tag{7}$$

Substituting from equation (6) into (3) we obtain the expansion of the coordinate functions for the geodesic in terms of its initial direction $p^i = u_s^i(0)$

$$u^i - u_0^i = p^i s - \Gamma_{kl}^i\,p^k p^l\,\frac{s^2}{2!} + \left[2\,\Gamma_{jn}^i\,\Gamma_{kl}^n - \frac{\partial\Gamma_{jk}^i}{\partial x^l}\right] p^j p^k p^l\,\frac{s^3}{3!} + \cdots \tag{8}$$

It is important to note that truncating the series expansion for the constrained geodesic renders the parameter ***s*** an approximation to the arclength rather than true arclength.

Likewise, the derivatives of a scalar function ***F*** with respect to arclength along a constrained geodesic are entirely expressible in terms of the initial direction of the geodesic or constrained geodesic.

Substituting the second and third order derivatives from (8) into the derivatives of ***F*** (7) we obtain

$$F_s = \left[ F_k \right] p^k \tag{9}$$

$$F_{ss} = \left[ F_{kl} - \Gamma^n_{kl} F_n \right] p^k p^l \tag{10}$$

$$F_{sss} = \left[ F_{klm} - 3 F_{kn} \Gamma^n_{lm} + F_n \left( 2 \Gamma^n_{kr} \Gamma^r_{lm} - \frac{\partial \Gamma^n_{kl}}{\partial u^m} \right) \right] p^k p^l p^m \tag{11}$$

For the geodesic in the embedding space the second order derivative of ***F*** with respect to arclength ***s*** in (10) is controlled by the matrix

$$M_{ij} = ( F_{ij} - \Gamma^k_{ij} F_k ) \tag{12}$$

For the constrained geodesic within the manifold this matrix is

$$\bar{M}_{ij} = ( F_{ij} - \Gamma^k_{ij} F_k ) - h^{pq} ( g^{kl} H^p_k F_l ) ( H^q_{ij} - \Gamma^k_{ij} H^q_k ) \tag{13}$$

Equation (12) is the Hessian Matrix of the scalar function ***F*** using absolute differentiation. Equation (13) shows that for the constrained geodesic from this there are to be subtracted the Hessians of the constraints multiplied by certain projection coefficients. The projection coefficients are found from the Gram system of order ***m***

$$h_{pq} \, \gamma_q = b_p \tag{14}$$

The Gram matrix and right hand side are defined by

$$h_{pq} = g^{kl} H^p_k H^q_l \tag{15}$$

$$b_p = g^{kl} H^p_k F_l \tag{16}$$

This is the usual *Gram* system for the projection coefficients of a vector onto a set of vectors, in this++-se for the projection of the gradient of the scalar function ***F*** onto the subspace of the constraints at the point ***P***.

Substituting the projection coefficients defined by the linear system (14) into (13) we may write

$$\bar{M}_{ij} = ( F_{ij} - \Gamma^k_{ij} F_k ) - \gamma_p ( H^p_{ij} - \Gamma^k_{ij} H^p_k ) \tag{17}$$

Equation (17) shows that the Hessian matrix controlling the variation of $F$ along a constrained geodesic $C$ is obtained from the Hessian for the unconstrained problem from which there are to be subtracted the Hessians of the constraints multiplied by certain projection coefficients. Each projection coefficient represents the projection of the normal to $F$ onto that constraint.

It is natural to investigate the quadratic approximation to $F$ on the manifold of the constraints. This may be done by considering the quadratic approximations to permissible constrained geodesics which constitute a quadratic approximation to the manifold at $P$ and to find the quadratic surface for $F$ defined on this manifold.

$$u^i - u_0^i = \left[p^i\right] s - \left[\left\{\Gamma_{kl}^i + h^{pq}(g^{in} H_n^p)(H_{kl}^q - \Gamma_{kl}^n H_n^q)\right\} p^k p^l\right] \frac{s^2}{2!} \tag{18}$$

$$F - F_0 = \left[F_k p^k\right] s + \left[\left\{(F_{kl} - \Gamma_{kl}^n F_n) - \gamma_p (H_{kl}^p - \Gamma_{kl}^n H_n^p)\right\} p^k p^l\right] \frac{s^2}{2!} \tag{19}$$

In the most common case where the embedding n-dimensional space is Euclidean these reduce to

$$u^i - u_0^i = \left[p^i\right] s - \left[(h^{pq}(g^{in} H_n^p) H_{kl}^q) p^k p^l\right] \frac{s^2}{2!} \tag{20}$$

$$F - F_0 = \left[F_k p^k\right] s + \left[(F_{kl} - \gamma_p H_{kl}^p) p^k p^l\right] \frac{s^2}{2!} \tag{21}$$

and the implied change of the metric to the trivial metric simplifies the Gram matrix and right hand side to

$$\begin{aligned} h_{pq} &= H_k^p H_k^q \\ b_p &= H_k^p F_k \end{aligned} \tag{22}$$

This leads us to consider the extremal directions of the matrix

$$\bar{M}_{ij} = F_{ij} - \gamma_p H_{ij}^p \tag{23}$$

subject to the conditions

$$H_k^p p^k = 0 \tag{24}$$

This is a problem from linear algebra which is solved using the block reflector [7]. The algorithm constructs the block reflector mapping the normals to the constraints surfaces $H_k^p \, (p = 1 \cdots m)$ into the vectors $\{e_{n-m} \cdots e_n\}$ and then applies the block reflector to the modified Hessian matrix (20). Next its principal minor of order $m$ is taken and its eigenvectors and eigenvalues computed. The eigenvectors are the extremal directions in the manifold and the eigenvalues are the curvatures. Finally the block reflector is used to map the eigenvectors back to the embedding n-dimensional space.

**GEODESIC COORDINATES**

The geodesic series expansion is closely related to geodesic polar coordinates which are introduced by setting

$$\xi^i = p^i s \tag{25}$$

equations (18) and (19) take the form

$$u^i - u_0^i = \xi^i - \tfrac{1}{2}\left[\Gamma^i_{kl} + h^{pq}(g^{in}H^p_n)(H^q_{kl} - \Gamma^n_{kl}H^q_n)\right]\xi^k\xi^l \tag{26}$$

$$F - F_0 = F_k\xi^k + \tfrac{1}{2}\left[(F_{kl} - \Gamma^n_{kl}F_n) - \gamma_p(H^p_{kl} - \Gamma^n_{kl}H^p_n)\right]\xi^k\xi^l \tag{27}$$

or using (45)

$$F - F_0 = F_k\xi^k + \tfrac{1}{2}\bar{M}_{kl}\,\xi^k\xi^l \tag{28}$$

subject to the constraints

$$H^p_k\xi^k = 0 \tag{29}$$

The transformation to geodesic coordinates (7) has as its Jacobian matrix

$$\frac{\partial u^i}{\partial \xi^j} = \delta^i_j - \left[\Gamma^i_{jk} + h^{pq}(g^{in}H^p_n)(H^q_{jk} - \Gamma^n_{jk}H^q_n)\right]\xi^k \tag{30}$$

If the embedding space is Euclidean this reduces to

$$\frac{\partial u^i}{\partial \xi^j} = \delta^i_j - \left[h^{pq}H^p_iH^q_{jk}\right]\xi^k \tag{31}$$

**MINIMUM OF F USING LAGRANGE MULTIPLIERS [INCORRECT]**

The minimum of $F$ on the quadratic surface spanned by the coordinates $\xi^i$ and subject to the constraints (29) is found by applying the method of Lagrange Multipliers by augmenting the objective function

$$G = F_k\xi^k + \tfrac{1}{2}\bar{M}_{kl}\,\xi^k\xi^l + \lambda_p H^p_k\xi^k \tag{32}$$

Differentiating the augmented objective function with respect to the $\xi^i$ and setting the result equal to zero we obtain

$$\bar{M}_{ik}\xi^k = -(F_i + \lambda_p H^p_i) \tag{33}$$

Therefore the solution can be written in the following form

$$\xi^i = p^i - \lambda_p q_p^i \tag{34}$$

where the vectors $p^i$ and $q_r^i$ are found by solving the linear systems of order $n$

$$\bar{M}_{ij} p_j = -F_i \tag{35}$$

$$\bar{M}_{ij} q_j^p = -H_i^p \qquad (p = 1 \cdots m) \tag{36}$$

The Lagrange multipliers are found by substituting (32) and (33) into (34) which leads to the following *non-symmetric* system of linear equations of order $m$

$$a_{pq} \lambda_q = b_p \tag{37}$$

$$a_{pq} = H_k^p q_q^k \tag{38}$$

$$b_p = H_k^p p_k \tag{39}$$

Finally we must normalize the search direction such that $s$ represents arclength

$$g_{mn} \xi^m \xi^n = 1 \tag{40}$$

Thus, the search direction implied by equations (75) and (76) together with the Lagrange multipliers (77) subject to the normalization condition (78) provides the solution to the constrained quadratic minimization problem for the stationary constrained search direction.